Euler-Maruyama scheme for SDEs with distributional drifts: linear and McKean-Vlasov

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SDEs with Low-regularity Coefficients: Theory and Numerics

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CONAHCYT
CONSEJO NACIONAL DE HUMANIDADES CIENCIAS Y TECNOLOGÍAS

Main aims

- Find the convergence rate of the Euler-Maruyama (EM) method to approximate solutions of SDEs with distributional coefficients
- Implement said numerical methods and compare the empirical and theoretical rate
- Study linear and McKean-Vlasov type SDEs with distributional coefficients

Reference

- **Preprint:** C. J., Issoglio, Palczewski. *Convergence rate of numerical scheme for SDEs with a distributional drift in Besov space.* arXiv:2309.11396 [8]
- **Implementation:** C. J., Issoglio, Palczewski. *Implementation of the Numerical Methods.* doi.org/10.5281/zenodo.8239606 [7]

Preprint

Implementation

Outline

Setting

We study the SDE

$$X_t = b(t, X_t)dt + dW_t$$
$$X_0 = x_0,$$

where the drift $b \in C([0, T]; C^{-\beta}(\mathbb{R}))$ for $0 < \beta < 1/2$, W_t is a one-dimensional Brownian motion and $x_0 \in \mathbb{R}$.

We concern ourselves with the theoretical analysis of numerical schemes and the implementation of said schemes.

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Theoretical results

- Existence of solutions is formulated through virtual solutions
- Two step approach:
  - Approximate the solution to the SDE X_t with a regularised SDE X_t^N
  - Create a numerical approximation $X_t^{N,m}$ of the regularised SDE
- Convergence rate $r(\beta) = \frac{\left(\frac{1}{2}-\beta\right)^2}{2\left(\frac{1}{2}-\beta\right)^2+\beta+1}$
  - $\lim_{\beta\downarrow 0} r(\beta) = \frac{1}{6}$
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Numerical implementation

- The trick of our implementation is on the definition of the drift
- We select the drift to be $b = \partial_x B \in C^{-\beta}$ for some function $B \in C^{1-\beta}$
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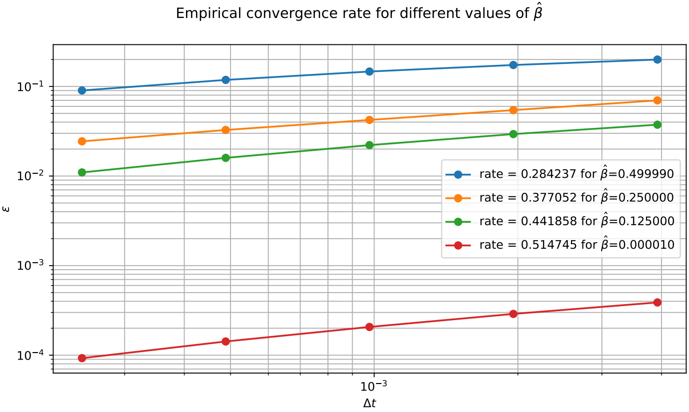
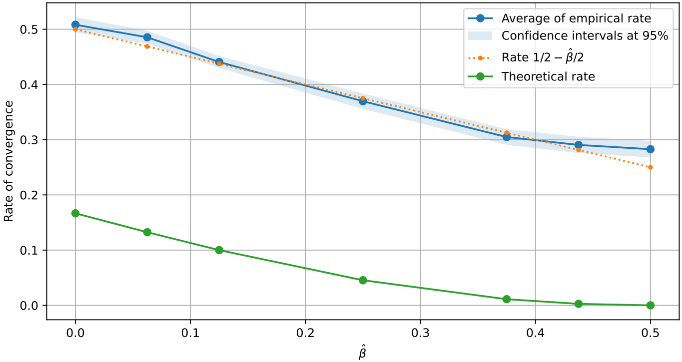
Convergence rate for the linear case



Empirical convergence rate for different values of $\hat{\beta}$

Theoretical, empirical and hypothetical rates



Comparison of the theoretical, empirical and hypothetical $1/2 - \hat{\beta}/2$ convergence rates for different values of $\hat{\beta}$

Outline

Setting

The McKean-Vlasov SDE (MVSDE) which concerns us is

$$\begin{aligned} dX_t &= F(v(t, X_t))b(t, X_t)dt + dW_t \\ X_0 &= x_0 \end{aligned}$$

Where F is a non-linear function, $v(t, \cdot)$ is the law density of X_t and finally b is as in the linear SDE

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The EM scheme

- Here we use an approach similar to the one for the linear case
- Instead of using propagation of chaos results
- It remains to find a way in which we compute the law density of the solution on each time step
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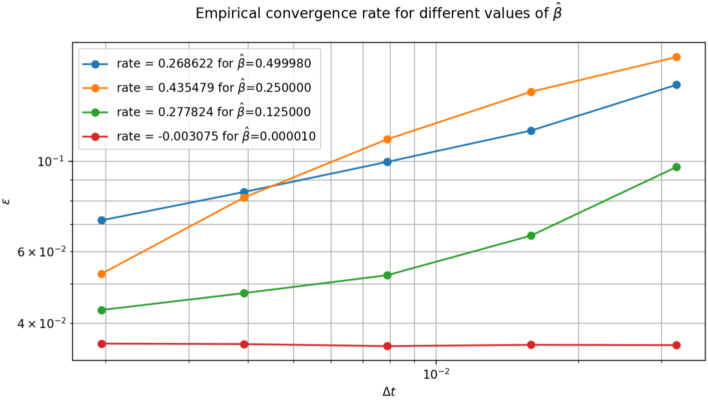
Convergence rate for the MVSDE



Comparison of the empirical convergence rate for McKean-Vlasov SDEs with respect to different $\hat{\beta}$

Outline

Future work

- Improve the method to compute the law within the EM scheme
- Find theoretical results for the MVSDE
- Improve the rate of convergence for the linear SDE*

Thank you!

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[1] Jianhai Bao and Xing Huang. *Approximations of Mckean-Vlasov SDEs with Irregular Coefficients*. June 4, 2019. DOI: 10.48550/arXiv.1905.08522. arXiv: 1905.08522 [math]. URL: http://arxiv.org/abs/1905.08522 (visited on 08/09/2022). preprint.

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